Abstract

In this note, we explain how \( f(a) = \log(1 - e^{-a}) = \log(1 - \exp(-a)) \) can be computed accurately, in a simple and optimal manner, building on the two related auxiliary functions \( \log1p(x) = \log(1 + x) \) and \( \expm1(x) = \exp(x) - 1 = e^x - 1 \). The cutoff, \( a_0 \), in use in \( R \) since 2004, is shown to be optimal both theoretically and empirically, using \( \text{Rmpfr} \) high precision arithmetic. As an aside, we also show how to compute \( \log(1 + e^x) \) accurately and efficiently.

Keywords: Accuracy, Cancellation Error, R, MPFR, Rmpfr.

1. Introduction: Not log() nor exp(), but log1p() and expm1()

In applied mathematics, it has been known for a very long time that direct computation of \( \log(1 + x) \) suffers from severe cancellation (in “1 + x”) whenever \( |x| \ll 1 \), and for that reason, we have provided \( \log1p(x) \) in \( R \), since \( R \) version 1.0.0 (released, Feb. 29, 2000). Similarly, \( \log1p() \) has been provided by C math libraries and has become part of C language standards around the same time, see, for example, IEEE and Open Group (2004).

Analogously, since \( R \) 1.5.0 (April 2002), the function \( \expm1(x) = \exp(x) - 1 = e^x - 1 \) accurately also for \( |x| \ll 1 \), where \( e^x \approx 1 \) is (partially) cancelled by “− 1”.

In both cases, a simple solution for small \( |x| \) is to use a few terms of the Taylor series, as

\[
\log1p(x) = \log(1 + x) = x - x^2/2 + x^3/3 - \ldots, \text{ for } |x| < 1, \tag{1}
\]

\[
\expm1(x) = \exp(x) - 1 = x + x^2/2! + x^3/3! + \ldots, \text{ for } |x| < 1, \tag{2}
\]

and \( n! \) denotes the factorial.

We have found, however, that in some situations, the use of \( \log1p() \) and \( \expm1() \) may not be sufficient to prevent loss of numerical accuracy. The topic of this note is to analyze the important case of computing \( \log (1 - e^x) = \log(1 - \exp(x)) \) for \( x < 0 \), computations needed in accurate computations of the beta, gamma, exponential, Weibull, t, logistic, geometric and hypergeometric distributions, and even for the logit link function in logistic regression. For the beta and gamma distributions, see, for example, DiDonato and Morris (1992)\(^1\), and further references mentioned in \( R \)'s \texttt{?pgamma} and \texttt{?pbeta} help pages. For the logistic distribution,

\(^1\)In the Fortran source, file “708”, also available as \texttt{http://www.netlib.org/toms/708}, the function ALNREL() computes \( \log1p() \) and REXP() computes \( \expm1() \).
Accurately Computing $\log(1 - \exp(-|a|))$ Assessed by the Rmpfr package

$F_L(x) = \frac{e^x}{1+e^x}$, the inverse, aka quantile function is $q_L(p) = \logit(p) := \log \frac{p}{1-p}$. If the argument $p$ is provided on the log scale, $\hat{p} := \log p$, hence $\hat{p} \leq 0$, we need

$$q\logis(\hat{p}, \log p = \text{TRUE}) = q_L(e^{\hat{p}}) = \logit(e^{\hat{p}}) = \log \frac{e^{\hat{p}}}{1 - e^{\hat{p}}} = \hat{p} - \log(1 - e^{\hat{p}}), \quad (3)$$

and the last term is exactly the topic of this note.

2. log1p() and expm1() for $\log(1 - \exp(x))$

Contrary to what one would expect, for computing $\log(1 - \exp(x)) = \log(1 - e^x)$ for $x < 0$, neither

$$\log(1 - \exp(x)) = \log(-\expm1(x)), \quad \text{nor} \tag{4}$$

$$\log(1 - \exp(x)) = \log1p(-\exp(x)), \quad \text{are uniformly sufficient for numerical evaluation. In (5), when x approaches 0, exp(x) approaches 1 and loses accuracy. In (4), when x is large, expm1(x) approaches -1 and similarly loses accuracy. Because of this, we will propose to use a function log1mexp(x) which uses either expm1 (4) or log1p (5), where appropriate. Already in R 1.9.0 (R Development Core Team (2004)), we have defined the macro R_D_LExp(x) to provide these two cases automatically.2

To investigate the accuracy losses empirically, we make use of the R package Rmpfr for arbitrarily accurate numerical computation, and use the following simple functions:

```r
R> library(Rmpfr)
R> t3.l1e <- function(a)
  { c(def = log(1 - exp(-a)), expm1 = log(-expm1(-a)), log1p = log1p(-exp(-a)))
  }
```

```r
R> ##' The relative Error of log1mexp computations:
R> relE.l1e <- function(a, precBits = 1024) {
  stopifnot(is.numeric(a), length(a) == 1, precBits > 50)
  da <- t3.l1e(a) ## double precision
  a. <- mpfr(a, precBits=precBits)
  ## high precision *and* using the correct case:
  mMa <- if(a <= log(2)) log(-expm1(-a.)) else log1p(-exp(-a.))
  structure(as.numeric(1 - da/mMa), names = names(da))
}
```

where the last one, relE.l1e() computes the relative error of three different ways to compute $\log(1 - \exp(-a))$ for positive $a$ (instead of computing $\log(1 - \exp(x))$ for negative $x$).

```
R> a.s <- 2^seq(-55, 10, length = 256)
R> ra.s <- t(sapply(a.s, relE.l1e))
R> cbind(a.s, ra.s) # comparison of the three approaches
   a.s   def    expm1    log1p
[1,] 2.7756e-17  -Inf  -7.9755e-17  -Inf
```

2look for “log(1-exp(x))” in http://svn.r-project.org/R/branches/R-1-9-patches/src/nmath/dpq.h
This is revealing: Neither method, log1p or expm1, is uniformly good enough. Note that for large \(a\), the relative errors evaluate to 1. This is because all three double precision methods give 0, and that is the best approximation in double precision (but not in higher \texttt{mpfr} precision), hence no problem at all, and we can restrict ourselves to smaller \(a\) (smaller than about 710, here).

What about really small \(a\)'s? Note here that

\begin{verbatim}
R> t3.l1e(1e-20)
    def expm1 log1p
          -Inf -46.052 -Inf
R> as.numeric(t3.l1e(mpfr(1e-20, 256)))
[1] -46.052 -46.052 -46.052
\end{verbatim}

both the default and the \texttt{log1p} method return \texttt{-Inf}, so, indeed, the \texttt{expm1} method is absolutely needed here.

Figure 1 visualizes the relative errors\(^3\) of the three methods. Note that the default basically gives the maximum of the two methods' errors, whereas the final \texttt{log1mexp()} function will have (approximately) minimal error of the two.

\begin{verbatim}
R> matplot(a.s, abs(ra.s), type = "l", log = "xy",
          col=cc, lty=lt, lwd=ll, xlab = "a", ylab = "", axes=FALSE)
R> legend("top", leg, col=cc, lty=lt, lwd=ll, bty="n")
R> draw.machEps <- function(alpha.f = 1/3, col = adjustcolor("black", alpha.f)) {
    abline(h = .Machine$double.eps, col=col, lty=3)
    axis(4, at=.Machine$double.eps, label=quote(epsilon[c]), las=1, col.axis=col)
}
R> eaxis(1); eaxis(2); draw.machEps(0.4)
\end{verbatim}

In Figure 2 below, we zoom into the region where all methods have about the same (good) accuracy. The region is the rectangle defined by the ranges of \(a\). and \(ra2\):

\begin{verbatim}
R> a. <- (1:400)/256
R> ra <- t(sapply(a., relE.l1e))
R> ra2 <- ra[,,-1]
\end{verbatim}

In addition to zooming in Figure 1, we want to smooth the two curves, using a method

\(^3\)Absolute value of relative errors, \[\left|\frac{(f(a) - f(a))/f(a)}{1 - f(a)/f(a)}\right| = \left|1 - f(a)/f(a)\right|,\] where \(f(a) = \text{log1mexp}(a)\) (7) is computed accurately by a 1024 bit \texttt{Rmpfr} computation
Accurately Computing $\log(1 - \exp(-|a|))$

Assessed by the \texttt{Rmpfr} package

\textbf{Figure 1:} Relative errors* of the default, $\log(1 - e^{-a})$, and the two methods “\texttt{expm1}” $\log(-\expm1(-a))$ and “\texttt{log1p}” $\log1p(-\exp(-a))$. Figure 2 will be a zoom into the gray rectangular region where all three curves are close.

assuming approximately normal errors. Notice however that neither the original, nor the log-transformed values have approximately symmetric errors, so we use \texttt{MASS::boxcox()} to determine the “correct” power transformation,

\begin{verbatim}
R> da <- cbind(a = a., as.data.frame(ra2))
R> library(MASS)
R> bc1 <- boxcox(abs(expm1) ~ a, data = da, lambda = seq(0,1, by=.01), plotit=.plot.BC)
R> bc2 <- boxcox(abs(log1p) ~ a, data = da, lambda = seq(0,1, by=.01), plotit=.plot.BC)
R> c(with(bc1, x[which.max(y)]),
    with(bc2, x[which.max(y)]))  # optimal powers
[1] 0.38 0.30
\end{verbatim}

i.e, the optimal boxcox exponent turns out to be close to $\frac{1}{3}$, which we use for smoothing in a “zoom-in” of Figure 1. Then, the crossover point of the two curves already suggests that the cutoff, $a_0 = \log 2$ is empirically very close to optimal.

\begin{verbatim}
R> matplot(a., abs(ra2), type = "l", log = "y", # ylim = c(-1,1)*1e-12,
          col=cc[-1], lwd=ll[-1], lty=lt[-1],
\end{verbatim}
ylim = yl, xlab = "a", ylab = "", axes=FALSE
R> legend("topleft", leg[-1], col=cc[-1], lwd=ll[-1], lty=lt[-1], bty="n")
R> eaxis(1); eaxis(2); draw.machEps()
R> lines(a., predict(s1)$y ^ 3, col=cc[2], lwd=2)
R> lines(a., predict(s2)$y ^ 3, col=cc[3], lwd=2)

\[
\log(\expm1(\exp(-a))) = \log(\exp(-\exp(-a)))
\]

Figure 2: A “zoom in” of Figure 1 showing the region where the two basic methods, “\texttt{expm1}” and “\texttt{log1p}” switch their optimality with respect to their relative errors. Both have small relative errors in this region, typically below \(\varepsilon_c := \texttt{Machine\$double.eps} = 2^{-52} \approx 2.22 \times 10^{-16}\). The smoothed curves indicate crossover close to \(a_0 := \log 2\).

Why is it very plausible to take \(a_0 := \log 2\) as approximately optimal cutoff? Already from Figure 2, empirically, an optimal cutoff \(a_0\) is around 0.7. We propose to compute

\[ f(a) = \log(1 - e^{-a}) = \log(1 - \exp(-a)), \quad a > 0, \] (6)

by a new method or function \texttt{log1mexp(a)}. It needs a cutoff \(a_0\) between choosing \texttt{expm1} for \(0 < a \leq a_0\) and \texttt{log1p} for \(a > a_0\), i.e.,

\[ f(a) = \log1mexp(a) := \begin{cases} 
\log(\expm1(-a)) & 0 < a \leq a_0 \ (:= \log 2 \approx 0.693) \\
\log1p(-\exp(-a)) & a > a_0.
\end{cases} \] (7)

The mathematical argument for choosing \(a_0\) is quite simple, at least informally: In which situations does \(1 - e^{-a}\) loose bits (binary digits) entirely independently of the computational algorithm? Well, as soon as it “spends” bits just to store its closeness to 1. And that is as soon as \(e^{-a} < \frac{1}{2} = 2^{-1}\), because then, at least one bit cancels. This however is equivalent to

\[ -a < \log(2^{-1}) = -\log(2) \text{ or } a > \log 2 =: a_0. \]
3. Computation of $\log(1+\exp(x))$

Related to $\log1mexp(a) = \log(1-e^{-a})$ is the log survival function of the logistic distribution $\log(1 - F_L(x)) = \log\frac{1}{1+e^{-x}} = -\log(1 + e^{x}) = -g(x)$, where

$$g(x) := \log(1 + e^{x}) = \log1p(e^{x}), \quad (8)$$

which has a “+” instead of a “−”, compared to $\log1mexp$, and is easier to analyze and compute, its only problem being large $x$’s where $e^{x}$ overflows numerically.\(^4\) As $g(x) = \log(1 + e^{x}) = \log(e^{x}(e^{-x} + 1)) = x + \log(1 + e^{-x})$, we see from (1) that

$$g(x) = x + \log(1 + e^{-x}) = x + e^{-x} + O((e^{-x})^{2}), \quad (9)$$

for $x \to \infty$. Note further, that for $x \to -\infty$, we can simplify $g(x) = \log(1 + e^{x})$ to $e^{x}$.

A simple picture quickly reveals how different approximations behave, where we have used `uniroot()` to determine the zero crossing, but will use slightly simpler cutoffs $x_0 = 37$, $x_1$ and $x_2$, in (10) below:

\[
\begin{align*}
R> \text{## Find } x_0, \text{ such that } \exp(x) &= g(x) \text{ for } x < x_0 : \\
R> f0 &\leftarrow \text{function}(x) \{ x \leftarrow \exp(x) - \log1p(\exp(x)) \} \\
R> x0 &\leftarrow \text{uniroot}(f0, c(-100, 0), \text{tol}=1e-13) \\
R> \text{str}(u0, \text{digits}=10) \\
\end{align*}
\]

List of 5

- \$root : num -36.39022698
- \$f.root : num 2.465190329e-32
- \$iter : int 81
- \$init.it : int NA
- \$estim.prec: num 7.815970093e-14

\[
\begin{align*}
R> x0 &\leftarrow u0[["root"]][[1]] \quad \text{## -36.39022698 --- note that } \approx \log(\\text{eps}_C) \\
R> \text{all.equal}(x0, -52.5 * \log(2), \text{tol}=1e-13) \\
\end{align*}
\]

[1] TRUE

\[
\begin{align*}
R> \text{## Find } x_1, \text{ such that } x + \exp(-x) &= g(x) \text{ for } x > x_1 : \\
R> f1 &\leftarrow \text{function}(x) \{ x \leftarrow (x + \exp(-x)) - \log1p(\exp(x)) \} \\
R> x1 &\leftarrow \text{uniroot}(f1, c(1, 20), \text{tol}=1e-13) \\
R> \text{str}(u1, \text{digits}=10) \\
\end{align*}
\]

List of 5

- \$root : num 16.40822612
- \$f.root : num 3.552713679e-15
- \$iter : int 18
- \$init.it : int NA
- \$estim.prec: num 5.684341886e-14

\[
\begin{align*}
R> x1 &\leftarrow u1[["root"]][[1]] \quad \text{## 16.408226} \\
R> \text{## Find } x_2, \text{ such that } x &= g(x) \text{ for } x > x_2 : \\
R> f2 &\leftarrow \text{function}(x) \{ x \leftarrow \log1p(\exp(x)) - x \} \\
R> x2 &\leftarrow \text{uniroot}(f2, c(5, 50), \text{tol}=1e-13) \\
R> \text{str}(u2, \text{digits}=10) \\
\end{align*}
\]

\(^4\)Indeed, for $x = 710$, $-g(x) = \log(1 - F_L(x)) = \text{plogis}(710, \text{lower}=\text{FALSE}, \log.p=\text{TRUE})$, underflowed to $-\text{Inf}$ in R versions before 2.15.1 (June 2012) from when on (10) has been used.
List of 5
$ root : num 33.2783501
$ f.root : num 7.105427358e-15
$ iter : int 9
$ init.it : int NA
$ estim.prec: num 6.394884622e-14

R> x2 <- u2["root"] ## 33.27835

R> par(mfcol= 1:2, mar = c(4.1,4.1,0.6,1.6), mgp = c(1.6, 0.75, 0))
R> curve(x+exp(-x) - log1p(exp(x)), 15, 25, n=2^11); abline(v=x1, lty=3)
R> curve(log1p(exp(x)) - x, 33.1, 33.5, n=2^10); abline(v=x2, lty=3)

Using double precision arithmetic, a fast and accurate computational method is to use

\[
g(x) = \begin{cases} 
\exp(x) & x \leq -37 \\
\log1p(\exp(x)) & -37 < x \leq x_1 := 18, \\
x + \exp(-x) & x_1 < x \leq x_2 := 33.3, \\
x & x > x_2, 
\end{cases}
\] (10)

where only the cutoff \( x_1 = 18 \) is important and the other cutoffs just save computations.\(^5\) Figure 3 visualizes the relative errors of the careless “default”, \( \log(1 + e^x) \), its straightforward correction \( \log1p(\exp(x)) \), the intermediate approximation \( x + e^{-x} \), and the large \( x \) (= \( x \)), i.e., the methods in (10), depicting that the (easy to remember) cutoffs \( x_1 \) and \( x_2 \) in (10) are valid. Note that the default method is fully accurate on this \( x \) range and only problematic when \( e^x \) begins to overflow, i.e., \( x > e_{\text{Max}} \), which is

\[
R> (eMax <- .Machine$double.max.exp * log(2))
\]

[1] 709.78
\[
R> exp(eMax * c(1, 1+1e-15))
\]

[1] 1.7977e+308 Inf

where we see that indeed \( e_{\text{Max}} = eMax \) is the maximal exponent without overflow.

4. Conclusion

We have used high precision arithmetic (R package \texttt{Rmpfr}) to empirically verify that com-

\(^5\)see plot \texttt{curve(log1p(exp(x)) - x, 33.1, 33.5, n=2^10)} above, revealing a somewhat fuzzy cutoff \( x_2 \).
Accurately Computing $\log(1 - \exp(-|a|))$ Assessed by the \textit{Rmpfr} package

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{Relative errors (via \textit{Rmpfr}, see footnote of Fig. 1) of four different ways to numerically compute $\log(1 + e^x)$. Vertical bars at $x_1 = 18$ and $x_2 = 33.3$ visualize the (2nd and 3rd) cutpoints of (10).}
\end{figure}

Computing $f(a) = \log(1 - e^{-a})$ is accomplished best via equation (7). In passing, we have also shown that accurate computation of $g(x) = \log(1 + e^x)$ can be achieved via (10). Note that a version of this note is available as vignette (in \textit{Sweave}, i.e., with complete R source) from the \textit{Rmpfr} package vignettes.

\section*{Session Information}
\begin{verbatim}
R> toLatex(sessionInfo(), locale=FALSE)
\end{verbatim}

- R version 3.5.2 Patched (2019-01-14 r75991), x86_64-pc-linux-gnu
- Running under: Fedora 28 (Twenty Eight)
- Matrix products: default
- Base packages: base, datasets, grDevices, graphics, methods, stats, utils
- Other packages: MASS 7.3-51.1, Rmpfr 0.7-2, gmp 0.5-13.2, polynom 1.3-9, sfsmisc 1.1-3
- Loaded via a namespace (and not attached): compiler 3.5.2, tools 3.5.2

\section*{References}


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