Characteristic Functions in the \textit{prob} package

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1 Introduction

The characteristic function (c.f.) of a random variable $X$ is defined by

$$\phi_X(t) = \mathbb{E}e^{itX}, \quad -\infty < t < \infty.$$  

When the distribution of $X$ is discrete with probability mass function (p.m.f.) $p_X(x)$, the c.f. takes the form

$$\phi_X(t) = \sum_{x \in S_X} e^{itx}p_X(x),$$

where $S_X$ is the support of $X$. When the distribution of $X$ is continuous with probability density function (p.d.f.) $f_X(x)$, the c.f. takes the form

$$\phi_X(t) = \int_{S_X} e^{itx}f_X(x)\,dx.$$

Characteristic functions have many, many useful properties: for example, every c.f. is uniformly continuous and bounded in modulus (by 1). Furthermore, a random variable has a distribution symmetric about 0 if and only if its associated c.f. is real-valued. For details, see [7].

Most of the below formulas came from [8, 9, 10]. Some of them involve special mathematical functions and a classical reference for them is [2], but many of the definitions have made it to Wikipedia (http://www.wikipedia.org/) and selected links to the respective Wikipedia topics have been listed when appropriate.

Note that the returned value of a characteristic function is a complex number, and is represented as such in R, even for those c.f.’s which correspond to symmetric distributions. Thus, \texttt{cfnorm(0) = 1 + 0i}, and \texttt{not cfnorm(0) = 1}. Depending on the application, the respective c.f.’s may need to be wrapped in \texttt{as.real()}. 

All of the below functions were written in straight R code; it would likely be possible to speed up evaluation if for example they were written in C or some other language. I would welcome any contributions for improvement in the \texttt{prob} package.

There are three special cases: the noncentral Beta, noncentral Student’s $t$, and Weibull distributions. For these the c.f.’s are integrated numerically and thus are subject to all of numerical integration’s limitations and idiosyncracies. I would be especially interested in and appreciative of a reference for these cases to be improved.

2 Characteristic functions

The formulas for all characteristic functions supported in the \texttt{prob} package are listed below, in alphabetical order of the function name.

2.1 Beta distribution: \texttt{cfbeta(t, shape1, shape2, ncp = 0)}

Let $\alpha$ and $\beta$ denote the \texttt{shape1} and \texttt{shape2} parameters, respectively. The p.d.f. is then

$$f_X(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1,$$

where $\Gamma$ is the gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1}e^{-u}\,du, \quad \alpha \neq 0, -1, -2, \ldots.$$

The characteristic function is given by

$$\phi_X(t) = _1F_1(\alpha; \alpha + \beta; it),$$
where \( _1F_1 \) is Kummer’s confluent hypergeometric function of the first kind, also known as Kummer’s \( M \), defined by

\[
_1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!},
\]

with \((a)_n = a(a + 1)(a + 2) \cdots (a + n - 1)\) the rising factorial. We calculate \( _1F_1 \) using \texttt{kummerM} in the \texttt{fAsianOptions} package.

As of the time of this writing, it seems that we must resort to calculating the characteristic function for the noncentral Beta by numerical integration according to the definition; see the source below. If you are aware of a way to more quickly/reliably calculate this c.f. with \texttt{R}, I would appreciate it if you would let me know.

**Source Code:**

```r
function (t, shape1, shape2, ncp = 0)
{
  if (shape1 <= 0 || shape2 <= 0)
    stop("shape1, shape2 must be positive")
  if (identical(all.equal(ncp, 0), TRUE)) {
    require(fAsianOptions)
    kummerM((0+1i) * t, shape1, shape1 + shape2)
  } else {
    fr <- function(x) cos(t * x) * dbeta(x, shape1, shape2, ncp)
    fi <- function(x) sin(t * x) * dbeta(x, shape1, shape2, ncp)
    Rp <- integrate(fr, lower = 0, upper = 1)$value
    Ip <- integrate(fi, lower = 0, upper = 1)$value
    return(Rp + (0+1i) * Ip)
  }
}
```

**2.2 Binomial distribution:** \texttt{cfbinom(t, size, prob)}

Let \( n \) and \( p \) denote the size and prob arguments, respectively. Then the p.m.f. is

\[
p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \ldots, n.
\]

The characteristic function is given by

\[
\phi_X(t) = \left[ p e^{it} + (1-p) \right]^n.
\]

**Source Code:**

```r
function (t, size, prob)
{
  if (size <= 0)
    stop("size must be positive")
  if (prob < 0 || prob > 1)
    stop("prob must be in [0,1]")
  (prob * exp((0+1i) * t) + (1 - prob)) ^ size
}
```
2.3 Cauchy Distribution: \texttt{cfcauchy(t, location = 0, scale = 1)}

Let $\theta$ and $\sigma$ denote the location and scale parameters, respectively. The p.d.f. is then

$$f_X(x) = \frac{1}{\pi \sigma} \frac{1}{1 + \left(\frac{x - \theta}{\sigma}\right)^2}, \quad -\infty < x < \infty.$$ 

The characteristic function is given by

$$\phi_X(t) = e^{it\theta - \sigma|t|}.$$

Source Code:

function (t, location = 0, scale = 1)
{
  if (scale <= 0)
    stop("scale must be positive")
  exp((0+1i) * location * t - scale * abs(t))
}
<environment: namespace:prob>

2.4 Chi-square Distribution: \texttt{cfchisq(t, df, ncp = 0)}

Let $p$ and $\delta$ denote the df and ncp parameters, respectively. The p.d.f. of the central chi-square distribution ($\delta = 0$) is then

$$f_X(x) = \frac{1}{\Gamma(p/2) \cdot 2^{p/2}} x^{p/2-1} e^{-x/2}, \quad x > 0.$$ 

One way to then write the p.d.f. of the noncentral chi-square distribution ($\delta > 0$) is with an infinite series:

$$f_X(x) = \sum_{k=0}^{\infty} \frac{e^{-\delta/2}(\delta/2)^k}{k!} f_{p+2k}(x), \quad x > 0,$$

where $f_{p+2k}$ is the p.d.f. of a central chi-square distribution with $p+2k$ degrees of freedom. The characteristic function in both cases is given by

$$\phi_X(t) = \frac{\exp \left\{ \frac{i\delta t}{1 - 2it} \right\}}{(1 - 2it)^{p/2}}.$$ 

Source Code:

function (t, df, ncp = 0)
{
  if (df < 0 || ncp < 0)
    stop("df and ncp must be nonnegative")
  exp((0+1i) * ncp * t/(1 - (0+2i) * t))/(1 - (0+2i) * t)^(df/2)
}
<environment: namespace:prob>

2.5 Exponential Distribution: \texttt{cfexp(t, rate = 1)}

This is the special case of the Gamma distribution when $\alpha = 1$. See Section 2.7.
Source Code:

```r
function (t, rate = 1) {
    cffgamma(t, shape = 1, scale = 1/rate)
}
```

<environment: namespace:prob>

2.6 $F$ Distribution: `cff(t, df1, df2, ncp, kmax = 10)`

Let $p$ and $q$ denote the $df1$ and $df2$ parameters, respectively, and let $\lambda$ denote the noncentrality parameter $ncp$. We may write the p.d.f. for the central $F$ distribution ($\lambda = 0$) with

$$f_X(x) = \frac{\Gamma((p + q)/2)}{\Gamma(p/2)\Gamma(q/2)} \left(\frac{p}{q}\right)^{p/2} x^{b/2-1} \left(1 + \frac{p}{q} x\right)^{-(p+q)/2}, \quad x > 0.$$ 

The characteristic function for central $F$ is given by

$$\phi_X(t) = \frac{\Gamma((p + q)/2)}{\Gamma(q/2)} \Psi\left(\frac{p}{2}, 1 - \frac{q}{2}; -\frac{qit}{p}\right),$$

where $\Psi$ is Kummer’s confluent hypergeometric function of the second kind, also known as Kummer’s $U$, defined by

$$\Psi(a, b; z) = \frac{\pi}{\sin \pi b} \left(\frac{1}{\Gamma(1 + a - b)\Gamma(b)} - z^{1-b}\frac{1}{\Gamma(a)\Gamma(2-b)}\right).$$

See [1] in the references. Kummer’s $U$ is calculated with `kummerU`, again from the `fAsianOptions` package.

The p.d.f. of the noncentral $F$ distribution ($\lambda \neq 0$) as

$$f_X(x) = f_{p,q}(x)e^{-\lambda/2}\sum_{k=0}^{\infty} \left\{\left(\frac{\lambda p}{q + px}\right)^k \frac{(p + q)(p + q + 2)\cdots(p + q + 2 \cdot k - 1)}{k! p(p + 2)\cdots(p + 2 \cdot k - 1)}\right\}, \quad x > 0,$$

where $f_{p,q}$ is the p.d.f. of the central $F$ distribution. The characteristic function for the noncentral $F$ distribution is given by

$$\phi_X(t) = e^{-\lambda/2}\sum_{k=0}^{\infty} \frac{(\lambda/2)^k}{k!} 1_F\left(p/2 + k; \frac{q}{2}, -qit/p\right),$$

where $1_F$ is Kummer’s confluent hypergeometric function of the first kind defined above; see Section 2.1. For the purposes of calculation, we may only use a finite sum to approximate the infinite series, thus the user should specify an upper value of $k$ to be used, denoted $kmax$, which has the default value of $kmax = 10$.

Source Code:

```r
function (t, df1, df2, ncp, kmax = 10) {
    if (df1 <= 0 || df2 <= 0)
        stop("df1 and df2 must be positive")
    require(fAsianOptions)
    if (identical(all.equal(ncp, 0), TRUE)) {
        gamma((df1 + df2)/2)/gamma(df2/2) * kummerU(-(0+1i) * df2 * t/df1, df1/2, 1 - df2/2)
    } else {
```
\[
\exp(-\text{ncp}/2) \ast \sum((\text{ncp}/2)\text{factorial}(0:\text{kmax}) \ast \\
\text{kummerM}(-\text{(0+1i)} \ast \text{df2} \ast \text{t/df1}, \text{df1/2} + 0:\text{kmax}, -\text{df2}/2))
\]

2.7 Gamma Distribution: \text{cfgamma}(t, \text{shape}, \text{rate} = 1, \text{scale} = 1/\text{rate})

Let \( \alpha \) and \( \beta \) denote the \text{shape} and \text{scale} parameters, respectively. The p.d.f. is then
\[
f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad x > 0.
\]
The characteristic function is given by
\[
\phi_X(t) = (1 - \beta i t)^{-\alpha}.
\]

Source Code:

\begin{verbatim}
function (t, shape, rate = 1, scale = 1/rate)
{
  if (rate <= 0 || scale <= 0)
    stop("rate must be positive")
  (1 - scale * (0+1i) * t)^(-shape)
}
\end{verbatim}

2.8 Geometric Distribution: \text{cfgeom}(t, \text{prob})

This is the special case of the Negative Binomial distribution when \( r = 1 \); see Section 2.12.

Source Code:

\begin{verbatim}
function (t, prob)
{
  cfnbinom(t, size = 1, prob = prob)
}
\end{verbatim}

2.9 Hypergeometric Distribution: \text{cfhyper}(t, m, n, k)

The p.m.f. takes the form
\[
p_X(x) = \binom{m}{x} \binom{n}{k-x} / \binom{m+n}{k}, \quad x = 0, \ldots, k; \quad x \leq m; \quad k-x \leq n.
\]
The characteristic function is given by
\[
\phi_X(t) = \frac{\binom{2F_1}{(k, -m; n-k+1; e^{it})}}{\binom{2F_1}{(-k, -m; n-k+1; 1)}},
\]
where \(\textstyle 2F_1\) is the Gaussian hypergeometric series defined by

\[
\textstyle 2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!},
\]

with \((a)_n\) the rising factorial defined as above in Section 2.1. See [3] in the References for details concerning \(\textstyle 2F_1\). We calculate it by means of the \texttt{hypergeo} function in the \texttt{hypergeo} package.

**Source Code:**

```r
function (t, m, n, k)
{
  if (m < 0 || n < 0 || k < 0)
    stop("m, n, k must be positive")
  hypergeo::hypergeo(-k, -m, n - k + 1, exp((0+1i) * t))/hypergeo::hypergeo(-k, -m, n - k + 1, 1)
}
```

2.10 Logistic Distribution: \texttt{cflogis(t, location = 0, scale = 1)}

Let \(\mu\) and \(\sigma\) denote the location and scale parameters, respectively. The p.d.f. is then

\[
f_X(x) = \frac{\text{e}^{-(x-\mu)/\sigma}}{\sigma (1 + \text{e}^{-(x-\mu)/\sigma})^2}, \quad -\infty < x < \infty.
\]

The characteristic function is given by

\[
\phi_X(t) = \text{e}^{i\mu t} \frac{\pi \sigma t}{\sinh(\pi \sigma t)},
\]

where

\[
\sinh(x) = \frac{\text{e}^x - \text{e}^{-x}}{2} = -i \sin ix,
\]


**Source Code:**

```r
function (t, location = 0, scale = 1)
{
  if (scale <= 0)
    stop("scale must be positive")
  ifelse(identical(all.equal(t, 0), TRUE), return(1), return(exp((0+1i) * location) * pi * scale * t/sinh(pi * scale * t)))
}
```

2.11 Lognormal Distribution: \texttt{cflnorm(t, meanlog = 0, sdlog = 1)}

Let \(\mu\) and \(\sigma\) denote the meanlog and sdlog parameters, respectively. The p.d.f. is then

\[
f_X(x) = \frac{1}{\sigma \sqrt{2\pi}} \frac{1}{x} \text{e}^{-(\ln x - \mu)^2/2\sigma^2}, \quad -\infty < x < \infty.
\]

The characteristic function is uniquely complicated and delicate. See [5] in the References. For fast numerical computation an algorithm due to Beaulieu is used, see [11].
2.12 Negative Binomial Distribution: `cfnbinom(t, size, prob, mu)`

Let \( r \) and \( p \) denote the size and prob parameters, respectively. We may write the p.m.f. as:

\[
p_X(x) = \binom{r + x - 1}{r - 1} p^r (1 - p)^x, \quad x = 0, 1, 2, \ldots
\]

The characteristic function is given by

\[
\phi_X(t) = \left( \frac{p}{1 - (1 - p)e^{it}} \right)^r.
\]

Source Code:

```r
function (t, size, prob, mu) {
  if (size <= 0)
    stop("size must be positive")
  if (prob <= 0 || prob > 1)
    stop("prob must be in (0,1]")
  if (!missing(mu)) {
    if (!missing(prob))
      stop("'prob' and 'mu' both specified")
    prob <- size/(size + mu)
  }
  (prob/(1 - (1 - prob) * exp((0+1i) * t)))^size
}
```

<environment: namespace:prob>
2.13 Normal Distribution: cfnorm(t, mean = 0, sd = 1)

Let $\mu$ and $\sigma$ denote the mean and sd parameters, respectively. The p.d.f. is

$$f_X(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty.$$ 

The characteristic function is given by

$$\phi_X(t) = e^{i\mu t + \sigma^2 t^2 / 2}.$$ 

Source Code:

```r
function (t, mean = 0, sd = 1) 
{ 
  if (sd <= 0)
    stop("sd must be positive")
  exp((0+1i) * mean - (sd * t)^2/2)
} 
<environment: namespace:prob>
```

2.14 Poisson Distribution: cfpois(t, lambda)

Let $\lambda$ denote the lambda parameter. The p.m.f. is

$$p_X(x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

The characteristic function is given by

$$\phi_X(t) = \exp \{\lambda(e^{it} - 1)\}.$$ 

Source Code:

```r
function (t, lambda) 
{ 
  if (lambda <= 0)
    stop("lambda must be positive")
  exp(lambda * (exp((0+1i) * t) - 1))
} 
<environment: namespace:prob>
```

2.15 Wilcoxon Signed Rank Distribution: cfsignrank(t, n)

See ?dsignrank for a discussion of the p.m.f. for this distribution; it is sufficient for our purposes to know that $f_X$ is supported on the integers $x = 0, 1, \ldots, n(n + 1)/2$. Since the support is finite, we may calculate the characteristic function according to the definition:

$$\phi_X(t) = \sum_{x=0}^{n(n+1)/2} e^{itx} f_X(x),$$

where $f_X$ is given by dsignrank().
Source Code:

```r
function (t, n)
{
    sum(exp(((0+1i) * t * 0:((n + 1) * n/2)) * dsignrank(0:((n + 1) * n/2), n))
}
<environment: namespace:prob>
```

### 2.16 Student’s $t$ Distribution: \texttt{cft(t, df, ncp)}

Let $p$ denote the \texttt{df} parameter. The p.d.f. is

$$f_X(x) = \frac{\Gamma[(p + 1)/2]}{\sqrt{p\pi}\Gamma(p/2)} \left( 1 + \frac{x^2}{p} \right)^{-(p+1)/2}, \quad -\infty < x < \infty.$$  

The formula used for the characteristic function was published by Hurst, see [12]. The characteristic function is given by

$$\phi_X(t) = \frac{K_{p/2}((\sqrt{p}|t|) \cdot (\sqrt{p}|t|)^{p/2}}{\Gamma(p/2)|2^{p/2-1}},$$

where $K_{\nu}$ is the modified Bessel Function of the second kind, defined by

$$K_{\nu}(x) = \pi I_{-\nu}(x) - I_{-\nu}(x),$$

and $I_{\alpha}$ is the modified Bessel Function of the first kind, defined by

$$I_{\alpha}(x) = i^{-\alpha} J_{\alpha}(ix),$$

with $J_{\alpha}(x)$ being a Bessel function of the first kind, defined by

$$J_{\alpha}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m + \alpha + 1)} \left( \frac{x}{2} \right)^{2m+\alpha}.$$  


As of the time of this writing, it seems that we must resort to calculating the characteristic function for the noncentral Student’s $t$ by numerical integration according to the definition; see the source below. If you are aware of a way to more quickly/reliably calculate this c.f. with \texttt{R}, I would appreciate it if you would let me know.

Source Code:

```r
function (t, df, ncp)
{
    if (missing(ncp))
        ncp <- 0
    if (df <= 0)
        stop("df must be positive")
    if (identical(all.equal(ncp, 0), TRUE)) {
        ifelse(identical(all.equal(t, 0), TRUE), 1 + (0+0i),
            as.complex(besselK(sqrt(df) * abs(t), df/2) * (sqrt(df) *
\[
\text{abs}(t))^{-(df/2)}/(\gamma(df/2) \cdot 2^{-(df/2 - 1)})
\]

else {
    fr <- function(x) cos(t * x) * dt(x, df, ncp)
    fi <- function(x) sin(t * x) * dt(x, df, ncp)
    Rp <- integrate(fr, lower = -Inf, upper = Inf)$value
    Ip <- integrate(fi, lower = -Inf, upper = Inf)$value
    return(Rp + (0+1i) * Ip)
}

<environment: namespace:prob>

2.17 Continuous Uniform Distribution: cfunif(t, min = 0, max = 1)

Let \(a\) and \(b\) denote the \(\text{min}\) and \(\text{max}\) parameters, respectively. The p.d.f. is

\[
f_X(x) = \frac{1}{b-a}, \quad a < x < b.
\]

The characteristic function is given by

\[
\phi_X(t) = \frac{e^{itb} - e^{ita}}{(b-a)it}.
\]

Source Code:

function (t, min = 0, max = 1)
{
    if (max < min)
        stop("min cannot be greater than max")
    ifelse(identical(all.equal(t, 0), TRUE), 1 + (0+0i), (exp((0+1i) *
        t * max) - exp((0+1i) * t * min))/((0+1i) * t * (max -
        min)))
}

<environment: namespace:prob>

2.18 Weibull Distribution: cfweibull(t, shape, scale = 1)

Let \(a\) and \(b\) denote the \text{shape} and \text{scale} parameters, respectively. The p.d.f. is

\[
f_X(x) = \frac{a}{b} \left(\frac{x}{b}\right)^{a-1} e^{-\left(x/b\right)^a}, \quad 0 < x < \infty.
\]

At the time of this writing, we must resort to calculating the characteristic function according to the definition; see the source below. If you know of a way to more quickly/reliably calculate this c.f. with R, I would appreciate it if you would let me know.

Source Code:

function (t, shape, scale = 1)
{
    if (shape <= 0 || scale <= 0)
        stop("shape and scale must be positive")
    fr <- function(x) cos(t * x) * dweibull(x, shape, scale)
\begin{verbatim}
fi <- function(x) sin(t * x) * dweibull(x, shape, scale)
Rp <- integrate(fr, lower = 0, upper = Inf)$value
Ip <- integrate(fi, lower = 0, upper = Inf)$value
return(Rp + (0+1i) * Ip)
}
\end{verbatim}

\section*{2.19 Wilcoxon Rank Sum Distribution: cfwilcox(t, m, n)}

See \texttt{?dwilcox} for a discussion of the p.m.f. for this distribution; it is sufficient for our purposes to know that $f_X$ is supported on the integers $x = 0, 1, \ldots, mn$. Since the support is finite, we may calculate the characteristic function according to the definition:

$$
\phi_X(t) = \sum_{x=0}^{mn} e^{itx} f_X(x),
$$

where $f_X$ is given by \texttt{dwilcox()}.

\textbf{Source Code:}

\begin{verbatim}
function (t, m, n)
{
    sum(exp((0+1i) * t * 0:(m * n)) * dwilcox(0:(m * n), m, n))
}
\end{verbatim}

\section*{3 R Session information}

\begin{verbatim}
> toLatex(sessionInfo())
\end{verbatim}

- R version 2.8.1 (2008-12-22), i486-pc-linux-gnu
- Locale: LC_CTYPE=en_US.UTF-8;LC_NUMERIC=C;LC_TIME=en_US.UTF-8;LC_COLLATE=en_US.UTF-8;LC_MONETARY=C;LC_MESSAGES=en_US.UTF-8;LC_PAPER=en_US.UTF-8;LC_NAME=C;LC_ADDRESS=C;LC_TELEPHONE=C;LC_MEASUREMENT=en_US.UTF-8;LC_IDENTIFICATION=C
- Base packages: base, datasets, graphics, grDevices, methods, stats, tcltk, utils
- Other packages: prob 0.9-2, svGUI 0.9-43, svMisc 0.9-45, svSocket 0.9-42

\section*{References}

[1] \url{http://en.wikipedia.org/wiki/Confluent_hypergeometric_function}


